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Low-temperature properties in the stochastic quantisation of the Brownian oscillator

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Abstract. The microscopic model of Ullersma for a harmonic oscillator in contact with a thermal reservoir is quantised in the framework of Nelson's stochastic mechanics. Eliminating the degrees of freedom of the thermal reservoir the stochastic process of the quantum Brownian oscillator is obtained as the sum of the thermal contribution and the quantum (zero-temperature) contribution. The properties of the quantum fluctuations are studied in detail and the equation of motion is derived, obtaining the first example of a non-Markovian Nelson process.

1. Introduction

The quantum treatment of a particle in contact with a thermal reservoir is an old problem (for a review see Dekker 1981) which is still the object of active research, due to the renewed interest originated in the pioneering work of Caldeira and Leggett (1983) on dissipative quantum tunnelling. Broadly speaking, in that problem one is interested in the interplay of quantum and thermal fluctuations in the decay of a metastable state. However, novel and interesting features have also been discovered in the simpler case of a harmonic oscillator or even for a free particle in a thermal reservoir (for a review and references see Grabert *et al* (1988)) at low temperature when the effect of the environment cannot be treated perturbatively.

Stimulated by these results, which have uncovered new richness and complexity in the physics of a quantum system interacting with the environment, in this paper we continue the study of the quantum Brownian oscillator from the point of view of Nelson's stochastic quantisation (for a review of previous work see Ruggiero and Zannetti (1985a)).

In the framework of stochastic mechanics (Nelson 1984, Guerra 1981) pure states of isolated quantum systems, i.e. wavefunctions, are associated with Markovian random processes. In this case randomness is intrinsic, since it is due only to the quantum nature of the system, and does not involve energy dissipation or irreversible phenomena.

In a previous series of papers Ruggiero and Zannetti (1985a) have extended the quantisation method of Nelson to systems interacting with a thermal environment. In that case the states of the system are still described by random processes, with the

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important difference that randomness is partly of intrinsic quantum origin, as for pure states, and partly is due to the elimination of the degrees of freedom of the environment, as in the classical treatment of Brownian motion. This structure was clearly demonstrated in an effort (Ruggiero and Zannetti 1983) to derive exactly the process of the Brownian oscillator on the basis of the microscopic model solved by Ford *et al* (1965). By quantising this model according to stochastic quantisation and then eliminating the degrees of freedom of the thermal bath, a random process was obtained as the sum of two contributions reflecting, respectively, thermal and quantum fluctuations. While the physical meaning of the thermal contribution is clear and carries the energy dissipation, which necessarily arises in the interaction with the infinitely many degrees of freedom of the reservoir, the quantum contribution is much more subtle. In fact, at zero temperature a novel situation arises. The large system as a whole (oscillator and thermal reservoir) is in the ground state where there are only zero-point fluctuations. Then, when the degrees of freedom of the reservoir are eliminated, while keeping the temperature zero, no dissipative processes can arise. Therefore in this case one expects to find a random process which, although radically different from those associated with pure quantum states (since in this case there is no wavefunction), still exhibits properties which are characteristic of the quantum fluctuations in pure states, such as energy conservation. In order to investigate in detail those properties in this paper we analyse the microscopic model of Ullersma (1966) for an oscillator in interaction with a heat bath according to the method of stochastic quantisation.

The paper is organised as follows. In § 2 the model is introduced and the main features of the exact solution which will be needed in the following are illustrated. Next, in § 3 Nelson's stochastic mechanics is summarised both for pure states and for states described by density matrices. Finally in § 4 the elimination of the degrees of freedom of the thermal reservoir is carried out, yielding the exact stochastic process executed by the Brownian oscillator at arbitrary temperature. The zero-temperature limit is analysed in detail and the conclusions are presented.

2. Microscopic model

In this section we summarise the main features of the microscopic model which are of interest in the following. We consider a system of $N+1$ particles with a quadratic Hamiltonian

$$H=\tfrac{1}{2}(\boldsymbol{p}\cdot\boldsymbol{p})+\tfrac{1}{2}\boldsymbol{q}\cdot V\boldsymbol{q} \tag{2.1}$$

where $\boldsymbol{q}=(q_0, q_1, \ldots, q_n)$ and $\boldsymbol{p}=(p_0, p_1, \ldots, p_n)$ are the position and momentum vectors for the entire system. In the model of Ullersma (1966) the interaction term is given by

$$\tfrac{1}{2}\boldsymbol{q}\cdot V\boldsymbol{q}=\sum_{n=0}^{N}\tfrac{1}{2}\omega_n^2 q_n^2+\sum_{n=1}^{N}\varepsilon_n q_n q_0 \tag{2.2}$$

which describes a harmonic oscillator (central or Brownian) with frequency ω_0, linearly coupled with a thermal bath constituted by the ensemble of the remaining N oscillators, with frequencies ω_n.

This model can be solved exactly in two different, but clearly equivalent, ways. In the original solution of Ullersma (Ullersma 1966, Riseborough *et al* 1985, Haake and

Reibold 1985) the Hamiltonian is diagonalised by introducing the eigenvectors of V:

$$V\boldsymbol{X}_\nu = z_\nu^2 \boldsymbol{X}_\nu. \tag{2.3}$$

Performing the canonical transformation

$$\begin{aligned} q_n &= \sum_\nu X_{n\nu} q'_\nu \\ p_n &= \sum_\nu X_{n\nu} p'_\nu \end{aligned} \tag{2.4}$$

where $X_{n\nu}$ are the components of $\boldsymbol{X}_\nu$, the normal-mode Hamiltonian is obtained:

$$H = \sum_\nu \tfrac{1}{2}(p_\nu'^2 + z_\nu^2 q_\nu'^2). \tag{2.5}$$

The $N+1$ eigenfrequencies z_ν are the positive zeros of the function

$$g(z) = z^2 - \omega_0^2 - \sum_{n=1}^{N} \frac{\varepsilon_n^2}{z^2 - \omega_n^2} \tag{2.6}$$

and the matrix elements $X_{n\nu}$ are given by

$$X_{0\nu}^2 = \left[\left(\frac{1}{2z}\frac{\mathrm{d}g(z)}{\mathrm{d}z}\right)_{z=z_\nu}\right]^{-1} \tag{2.7}$$

$$X_{n\nu} = \frac{\varepsilon_n}{z^2 - \omega_n^2} X_{0\nu}. \tag{2.8}$$

The eigenvalues in (2.3) need to be positive for stability and this requires the following condition on the parameters of the model:

$$\omega_0^2 - \sum_{n=1}^{N} \frac{\varepsilon_n^2}{\omega_n^2} \geqslant 0. \tag{2.9}$$

Solving the initial-value problem for the normal modes and inserting into (2.4), the exact solution for the central oscillator is found:

$$p_0(t) = \dot{q}_0(t) \tag{2.10}$$

$$q_0(t) = \sum_{n=0}^{N} [\dot{A}_n(t) q_n(0) + A_n(t) p_n(0)] \tag{2.11}$$

with

$$A_n(t) = \sum_\nu X_{0\nu} X_{n\nu} z_\nu^{-1} \sin(z_\nu t) \tag{2.12}$$

and where $\{q_n(0), p_n(0)\}$ is a set of initial conditions.

Alternatively (Eckhardt 1987, Ford and Kac 1987), rather than going to the normal modes, the particular structure of the interaction matrix (2.2) yields canonical equations of motion of the following form:

$$\begin{aligned} \dot{q}_0 &= p_0 \\ \dot{p}_0 &= -\omega_0^2 q_0 - \sum_{n=1}^{N} \varepsilon_n q_n \end{aligned} \tag{2.13}$$

$$\begin{aligned} \dot{q}_n &= p_n \\ \dot{p}_n &= -\omega_n^2 q_n - \varepsilon_n q_0 \qquad n \neq 0. \end{aligned} \tag{2.14}$$

Solving (2.14) and inserting into (2.13) the degrees of freedom of the thermal reservoir are eliminated and one obtains an equation of motion for the central oscillator alone:

$$\dot{q}_0(t) = p_0(t)$$
$$\dot{p}_0(t) = -\omega_0^2 q_0(t) + B(0)q_0(t) - \int_0^t \mathrm{d}s\, B(t-s)\dot{q}_0(s) - B(t)q_0(0) + F(t) \tag{2.15}$$

where

$$B(t) = \sum_{n=1}^{N} \frac{\varepsilon_n^2}{\omega_n^2} \cos(\omega_n t) \tag{2.16}$$

and

$$F(t) = -\sum_{n=1}^{N} [\varepsilon_n q_n(0)\cos(\omega_n t) + (p_n(0)/\omega_n)\sin(\omega_n t)]. \tag{2.17}$$

The set of equations (2.10)-(2.12) and (2.15)-(2.17) are equivalent. The quantum solution is obtained from the above formulae, regarding $q_n(t)$ and $p_n(t)$ as operators in the Heisenberg representation. Considering an initial condition specified by thermal equilibrium for the entire system, namely with expectations for the initial values given by the canonical distribution

$$\langle q_n(0)q_m(0)\rangle = \frac{1}{\omega_n^2}\langle p_n(0)p_m(0)\rangle$$
$$= \delta_{nm}\frac{1}{2\omega_n}\hbar \coth(\beta\hbar\omega_n/2) \tag{2.18}$$

$$\langle q_n(0)p_m(0)\rangle = -\langle p_m(0)q_n(0)\rangle = \tfrac{1}{2}\mathrm{i}\hbar\,\delta_{nm} \tag{2.19}$$

(2.15) becomes the quantum Langevin equation, where the operator-valued random force $F(t)$ has the expectations

$$\langle F(t)\rangle = 0 \tag{2.20}$$

$$\tfrac{1}{2}\langle F(0)F(t) + F(t)F(0)\rangle = \sum_{n=1}^{N}\frac{\varepsilon_n^2}{\omega_n^2} E(\omega_n, T)\cos(\omega_n t) \tag{2.21}$$

with

$$E(\omega_n, T) = \tfrac{1}{2}\hbar\omega_n \coth(\tfrac{1}{2}\beta\hbar\omega_n). \tag{2.22}$$

In the following we will need the time-ordered correlation function

$$W(t) = \langle T[q_0(t)q_0(0)]\rangle \tag{2.23}$$

where T is the time-ordering operator, given by

$$W(t) = W_\beta(t) + W_Q(t) \tag{2.24}$$

with

$$W_\beta(t) = \frac{1}{\pi\mathrm{i}}\oint \mathrm{d}z \frac{P(z,\beta)}{zg(z)}\cos(zt) \tag{2.25}$$

$$P(z,\beta) = \frac{\hbar z}{\exp(\beta\hbar z) - 1} \tag{2.26}$$

$$W_Q(t) = \frac{\hbar}{2\pi\mathrm{i}}\oint \mathrm{d}z \frac{\exp(-\mathrm{i}zt)}{g(z)} \tag{2.27}$$

and where the contour of integration encircles the positive zeros of $g(z)$.

The latter contribution describes the correlations of the zero-point fluctuations, while $W_\beta(t)$ contains the contribution of the thermal fluctuations and vanishes at zero temperature ($\beta \to \infty$).

So far we have considered a bath of N oscillators, with N finite, and the results obtained are exact. Next, in order to introduce irreversibility effects, the limit of a large number of oscillators is considered by introducing a continuum of frequencies characterised by the spectral strength function

$$\gamma(\omega)\Delta\omega = \sum_{\omega<\omega_n<\omega+\Delta\omega} \varepsilon_n^2. \tag{2.28}$$

Replacing the sum by an integral the positivity condition (2.9) and (2.16) becomes

$$\omega_0^2 - \int_0^\infty d\omega \frac{\gamma(\omega)}{\omega^2} \geqslant 0 \tag{2.29}$$

and

$$B(t) = \int_0^\infty d\omega \frac{\gamma(\omega)}{\omega^2} \cos(\omega t). \tag{2.30}$$

From (2.29) we see that the positivity condition can be satisfied only if $\omega^{-2}\gamma(\omega)$ vanishes for ω large. Following Ullersma we can then assume that the spectral strength function, without violating (2.29), is such that

$$\omega^{-2}\gamma(\omega) = \omega_0^{-2}\gamma(\omega_0) = 4\Gamma/\pi \tag{2.31}$$

over a region of size $\omega_B \gg \omega_0$, where $\omega_B^{-1} = \tau_B$ is a response time characteristic of the thermal bath, much shorter than the characteristic time of the central oscillator.

Then, for times large with respect to τ_B, (2.30) can be approximated by

$$B(t) \simeq 4\Gamma\delta(t) \tag{2.32}$$

and inserting into (2.15) the quantum Langevin equation takes the more familiar form (Riseborough *et al* 1985)

$$\ddot{q}_0(t) = -\Omega_1^2 q_0(t) - 2\Gamma\dot{q}_0(t) + F(t) \tag{2.33}$$

where

$$\Omega_1^2 = \omega_0^2 - B(0) = \omega_0^2 - \int_0^\infty d\omega \frac{\gamma(\omega)}{\omega^2} \tag{2.34}$$

and the autocorrelation of the stochastic force $F(t)$ takes the form

$$\tfrac{1}{2}\langle F(t)F(0) + F(0)F(t)\rangle = \int_0^\infty d\omega \frac{\gamma(\omega)}{\omega^2} E(\omega,\beta)\cos(\omega t). \tag{2.35}$$

In this approximation the following expressions for the correlation functions introduced above are obtained:

$$W_\beta(t) = \frac{4\Gamma}{\pi}\int_0^\infty d\omega \frac{P(\omega,\beta)\cos(\omega t)}{[(\omega^2-\Omega_1^2)^2 + 4\Gamma^2\omega^2]} \tag{2.36}$$

$$W_Q(t) = \frac{4\Gamma}{\pi}\int_0^\infty d\omega \frac{\hbar\omega \exp(-i\omega|t|)}{2[(\omega^2-\Omega_1^2)^2 + 4\Gamma^2\omega^2]}. \tag{2.37}$$

3. Stochastic quantisation

In the next section a description of the quantum Brownian oscillator in terms of classical random processes will be derived. As a preliminary step, in this section we present stochastic mechanics for pure and mixed states.

According to Nelson's scheme of quantisation (Nelson 1984, Guerra 1981) a particle in a pure quantum state, with wavefunction $\psi(x, t)$, is described by a classical Markovian random process $x(t)$ obeying a stochastic differential equation of the form

$$\dot{x}(t) = b(x, t) + \theta(t) \tag{3.1}$$

where $\theta(t)$ is a Gaussian white noise with expectations (here we take the mass $m = 1$)

$$\langle \theta(t) \rangle = 0 \tag{3.2}$$

$$\langle (\theta(t)\theta(t')) \rangle = \hbar\delta(t - t') \tag{3.3}$$

and the drift $b(x, t)$ is related to the wavefunction by

$$b(x, t) = \tfrac{1}{2}\hbar \frac{\partial}{\partial x} \ln \rho(x, t) + \hbar \frac{\partial}{\partial x} S(x, t) \tag{3.4}$$

where

$$\rho(x, t) = |\psi(x, t)|^2 \tag{3.5}$$

$$S(x, t) = \hbar \operatorname{Im}(\ln \psi(x, t)). \tag{3.6}$$

As an explicit example, useful for the following, let us consider the coherent states $\psi_\alpha(x, t)$ of an oscillator with Hamiltonian $H = \frac{1}{2}(p^2 + \omega^2 q^2)$.

It has been shown elsewhere (Guerra and Loffredo 1981, Ruggiero and Zannetti 1982) that the stochastic process $x_\alpha(t)$ associated with $\psi_\alpha(x, t)$ can be written as the sum

$$x_\alpha(t) = q_\alpha(t) + \xi(t) \tag{3.7}$$

where $q_\alpha(t)$ is the solution of the classical equations of motion

$$\begin{aligned} \dot{q} &= p \\ \dot{p} &= -\omega^2 q \end{aligned} \tag{3.8}$$

with initial conditions $(q(0), p(0))$ related to the complex label α of the coherent state by

$$\alpha = (\omega q(0) + \mathrm{i} p(0))/(2\hbar\omega)^{1/2} \tag{3.9}$$

and $\xi(t)$ is the stationary stochastic process associated with the ground state of the oscillator with the stochastic differential equation

$$\dot{\xi}(t) = -\omega\xi(t) + \theta(t). \tag{3.10}$$

Clearly (3.10) defines a Gaussian process with zero mean and autocorrelation function

$$\langle \xi(0)\xi(t) \rangle = \frac{\hbar}{2\omega} \exp(-\omega|t|). \tag{3.11}$$

The next step is to extend stochastic mechanics to impure states (Ruggiero and Zannetti 1983). It is convenient to use the description of the system (2.5) in terms of normal modes and to introduce the coherent-state basis for each normal mode $\{\psi_{\alpha_\nu}(x, t)\}$. The density matrix associated with the canonical ensemble then takes the product form

$$\rho = \frac{\exp(-\beta H)}{\operatorname{Tr}\exp(-\beta H)} = \prod_\nu \int \mathrm{d}q'_\nu(0)\, \mathrm{d}p'_\nu(0) Q[q'_\nu(0), p'_\nu(0)] |\psi_{\alpha_\nu}\rangle\langle\psi_{\alpha_\nu}| \tag{3.12}$$

where α_ν is defined in (3.9) and the weight function is given by

$$Q[q'_\nu(0), p'_\nu(0)] = \frac{c_\nu}{2\pi\hbar} \exp\left[-\frac{c_\nu}{\hbar}\left(z_\nu q'^2_\nu(0) + \frac{1}{z_\nu} p'^2_\nu(0)\right)\right] \tag{3.13}$$

with

$$c_\nu = \exp(\beta\hbar z_\nu) - 1. \tag{3.14}$$

Since we know how to construct the random processes associated with each coherent state, the stochastic description of the density matrix (3.12) is readily obtained by introducing a set of random processes $\{x'_\nu(t)\}$ of the form (3.7), one for each normal mode:

$$x'_\nu(t) = q'_\nu(t) + \xi'_\nu(t) \tag{3.15}$$

where $q'_\nu(t)$ is the classical normal-mode coordinate and $\xi'_\nu(t)$ is a stationary random process obeying an equation of motion of the type (3.10) with frequency z_ν and white noise $\theta_\nu(t)$ satisfying

$$\langle\theta_\nu(t)\rangle = 0 \tag{3.16}$$

$$\langle\theta_\nu(t)\theta_\mu(t')\rangle = \hbar\delta_{\nu\mu}\delta(t-t'). \tag{3.17}$$

The connection between the multidimensional stochastic process $\{x'_\nu(t)\}$ and the density matrix (3.12) is obtained, assuming that the initial conditions of $q'_\nu(t)$ are distributed according to the weight function (3.13).

4. Quantum Brownian oscillator

The set of stochastic processes (3.15) derived in the previous section give a detailed microscopic description of the equilibrium states for the entire system. It must be emphasised that, although given in terms of random processes, this description is time reversible and energy conserving.

The programme now is to reduce the description, by elimination of the degrees of freedom of the thermal reservoir, arriving at an effective stochastic process for the central oscillator alone. In the limit of a large reservoir, dissipation and time irreversibility are expected to arise.

First notice that $q'_\nu(t)$ and $\xi'_\nu(t)$ in (3.15) are independent objects. Therefore the outcome of the reduction is expected to be a stochastic process for the central oscillator of the form

$$x_0(t) = q_0(t) + \xi_0(t) \tag{4.1}$$

where $q_0(t)$ and $\xi_0(t)$ are themselves independent random processes obtained, respectively, by carrying out the reduction among the classical components $\{q'_\nu\}$ and the quantum components $\{\xi'_\nu\}$ in (3.15). Furthermore, from the linearity of the system it follows that $x_0(t)$, as well as its components $q_0(t)$ and $\xi_0(t)$, is a Gaussian process. Therefore all the information is contained in the mean, which in this case vanishes:

$$\langle x_0(t)\rangle = \langle q_0(t)\rangle = \langle \xi_0(t)\rangle = 0 \tag{4.2}$$

and in the pair correlation functions:

$$\begin{aligned} S_x(t) &= \langle x_0(0)x_0(t)\rangle \\ S_q(t) &= \langle q_0(0)q_0(t)\rangle \\ S_\xi(t) &= \langle \xi_0(0)\xi_0(t)\rangle \end{aligned} \tag{4.3}$$

related by

$$S_x(t) = S_q(t) + S_\xi(t). \tag{4.4}$$

Let us first consider the thermal component $q_0(t)$. Since the normal modes q'_ν obey the classical equations of motion, by the same arguments presented in § 2, it follows that $q_0(t)$ is driven by an equation of motion of the form (2.15). Keeping in mind that the initial values appearing in the random force (2.17) are now distributed according to (3.13), for the thermal correlation function we find

$$\begin{aligned} S_q(t) &= \sum_\nu X_{0\nu}^2 P(z_\nu, \beta) z_\nu^{-2} \cos(z_\nu t) \\ &= \frac{1}{2\pi \mathrm{i}} \oint \mathrm{d}z \frac{P(z,\beta)}{zg(z)} \cos(zt) \end{aligned} \tag{4.5}$$

and comparing with (2.25)

$$S_q(t) = W_\beta(t). \tag{4.6}$$

Stochastic quantisation and standard methods give the same result for the contribution of the thermal fluctuations.

Next we consider the zero-temperature behaviour. In this case the entire system is in the ground state and, as a whole, undergoes zero-point fluctuations. The problem we are now addressing is the description of the fluctuations of the central oscillator as the degrees of freedom of the thermal reservoir are eliminated, while keeping the system at zero temperature.

The stochastic process of the Brownian oscillator is related to the corresponding ground-state processes in the normal modes by

$$\xi_0(t) = \sum_\nu X_{0\nu} \xi'_\nu(t) \tag{4.7}$$

which yields the following expression for the correlation function:

$$\begin{aligned} S_\xi(t) &= \sum_\nu X_{0\nu}^2 \frac{\hbar}{2z_\nu} \exp(-z_\nu |t|) \\ &= \frac{1}{2\pi \mathrm{i}} \oint \mathrm{d}z \frac{\exp(-z_\nu t)}{g(z)} \end{aligned} \tag{4.8}$$

where the contour of integration goes around the positive zeros of $g(z)$.

As a first observation, notice that comparing with (2.27) one finds

$$S_\xi(t) = W_Q(-\mathrm{i}t) \tag{4.9}$$

namely the ground-state correlation function, obtained through stochastic quantisation, *coincides* with the Euclidean version (Schwinger function) of the corresponding time-ordered correlation function (Wightman function), *exactly as in the case of non-dissipative systems* (Guerra and Ruggiero 1973).

The above result has been derived without going to the limit of a large reservoir. However it is clear that (4.9) will also hold as the limit of large N is taken. Therefore, the elimination of the degrees of freedom of the reservoir, provided the system is kept at zero temperature, does not introduce a mechanism for dissipation, contrary to what occurs at finite temperature.

The above statement can be clarified considering the response function of the Brownian oscillator:

$$\chi(\omega) = -\sum_{\nu} \frac{X_{0\nu}^2}{\omega^2 - z_\nu^2} \tag{4.10}$$

which is related to the Fourier transform of $S_\xi(t)$ by

$$S_\xi(\omega) = \hbar\chi'(\mathrm{i}\omega) \tag{4.11}$$

where $\chi'(z)$ is the real part of $\chi(z)$. Again, this is the same relation which holds for a single particle in a pure quantum state (Ruggiero and Zannetti 1985b), where fluctuations are only of quantum origin and clearly not related to dissipation. The physical meaning of (4.11) is readily understood considering a classical oscillator of frequency ω_0, subjected to an external force of frequency ω. This is a non-dissipative system, coupled to an external energy source, whose average energy per cycle $\bar{E}$ does not change with time and is related to the average square displacement by

$$\bar{x}^2 = 2\bar{E}\chi'(\mathrm{i}\omega) \tag{4.12}$$

where

$$\chi'(\omega) = \frac{1}{\omega_0^2 - \omega^2} \tag{4.13}$$

is the real part of the response function of the *undamped* harmonic oscillator.

Rewriting (4.11) as

$$\omega S_\xi(\omega) = 2(\hbar\omega/2)\chi'(\mathrm{i}\omega) \tag{4.14}$$

the similarity with (4.12) is evident, after replacing the average stored energy $\bar{E}$ with the zero-point energy $\hbar\omega/2$. Hence, the fluctuation-response relation (4.11) is the signature of the non-dissipative nature of the dynamics associated with the stochastic process $\xi_0(t)$. We emphasise that, if this can be naturally expected for a Nelson process in a pure quantum state (i.e. a wavefunction), it is a remarkable result in the present case, where the process $\xi_0(t)$ is obtained through an elimination of degrees of freedom.

The next step in the study of the random process $\xi_0(t)$ is the derivation of the equation of motion. We already know, from the stochastic mechanics of pure states, that the isolated oscillator in its ground state performs a Markovian random process and obeys the equation of motion (3.10). From the above discussion we also know that, when the thermal bath is introduced, keeping the system in the ground state, the physical properties of the fluctuations in some sense are not changed. However we expect the equation of motion to be strongly modified by the thermal bath. In particular we expect the bath to introduce memory effects, spoiling Markovianity, and yielding an equation of motion of the general form

$$\dot{\xi}(t) = -\int_0^t \mathrm{d}s\, L(t-s)\xi(s) + \eta(t) \qquad t > t_0 \tag{4.15}$$

where $L(t)$ is a memory kernel and $\eta(t)$ is a random noise with zero mean and autocorrelation function

$$\langle \eta(t)\eta(t')\rangle = f(t-t') \tag{4.16}$$

to be determined.

Since, as remarked above, the random process $\xi_0(t)$ is Gaussian and all information is contained in the correlation function $S_\xi(t)$, we must extract the unknown functions $L(t)$ and $f(t)$ from (4.8). From (4.15) it is easy to derive the equation of motion for $S_\xi(t)$:

$$\frac{\mathrm{d}S_\xi(t)}{\mathrm{d}t} = -\int_0^t \mathrm{d}s\, L(t-s)S_\xi(s) + R(t) \tag{4.17}$$

where the quantity

$$R(t) \equiv \langle \xi(0)\eta(t)\rangle \tag{4.18}$$

in turn obeys the equation of motion

$$\frac{\mathrm{d}R(t)}{\mathrm{d}t} = -\int_t^\infty \mathrm{d}s\, L(s-t)R(s) - f(t). \tag{4.19}$$

Laplace transforming, we find

$$\tilde{S}_\xi(z) = \frac{\tilde{R}(z) + S_\xi(0)}{z + \tilde{L}(z)} \tag{4.20}$$

$$\tilde{R}(z) = \frac{R(0) - \tilde{f}(z)}{z - \tilde{L}(-z)} \tag{4.21}$$

On the other hand, from (4.8) the Laplace transform of $S_\xi(t)$ is given by

$$\tilde{S}_\xi(z) = \frac{\hbar}{2\Omega(z)} \frac{1}{z + \Omega(z)} \tag{4.22}$$

after rewriting (2.6) as

$$g(z) = z^2 - \Omega^2(z) \tag{4.23}$$

with

$$\Omega^2(z) \equiv \omega_0^2 + \sum_{n=1}^{N} \frac{\varepsilon_n^2}{z^2 - \omega_n^2}. \tag{4.24}$$

Comparing (4.20) and (4.22) one can identify

$$\tilde{L}(z) = \Omega(z) \tag{4.25}$$

$$\tilde{R}(z) + S_\xi(0) = \hbar/2\Omega(z) \tag{4.26}$$

and from the latter relation the Laplace transform of the noise autocorrelation function is given by

$$\tilde{f}(z) = (S_\xi(0) - \hbar/2\Omega(z))(z - \Omega(-z)) + R(0). \tag{4.27}$$

Equations (4.25) and (4.27) give the dependence on the thermal bath of both the memory kernel and the random noise which, in the general case, are quite complicated functions. Furthermore, one can see that the dissipative finite-temperature kernel $B(t)$ and the non-dissipative zero-termperature kernel $L(t)$ are related by

$$\tilde{L}^2(z)=\omega_0^2+\mathrm{i}z\tilde{B}(\mathrm{i}z)-B(0) \tag{4.28}$$

using (4.24) and the relation

$$z\tilde{B}(z)-B(0)=-\sum_{n=1}^{N}\frac{\varepsilon_n^2}{z^2+\omega_n^2}. \tag{4.29}$$

In order to gain some insight into the result, let us first recover the Markovian limit of the free oscillator. Setting all the coupling constants to zero in (4.24) and recalling that in this case $S_\xi(0)=\hbar/2\omega_0$ and $R(0)=\hbar/2$, from (4.25) and (4.27) it follows that

$$\begin{aligned} L(t-t')&=\omega_0\delta(t-t')\\ f(t-t')&=\hbar\delta(t-t'). \end{aligned} \tag{4.30}$$

A less straightforward Markovian limit is obtained, neglecting the z dependence of the generalised frequency $\Omega(z)$ in (4.24) and obtaining the renormalised frequency

$$\Omega_1^2=\omega_0^2-\sum_{n=1}^{N}\frac{\varepsilon_n^2}{\omega_n^2} \tag{4.31}$$

which coincides with (2.34) in the limit of a large reservoir. In this case one obtains the exponentially decaying correlation function

$$S_\xi(t)=\frac{2\hbar}{\Omega_1}\exp(-\Omega_1 t). \tag{4.32}$$

This limit is more interesting since Markovian behaviour is recovered without throwing away the thermal bath. The physical meaning is more clearly understood in the limit of a large reservoir, where (4.31) coincides with (2.34) and (4.32) is obtained from the exact expression of the correlation function:

$$S_\xi(t)=\frac{4\Gamma}{\pi}\int_0^\infty \mathrm{d}\omega\frac{\hbar\omega\exp(-\omega|t|)}{2[(\omega^2-\Omega_1^2)^2+4\Gamma^2\omega^2]} \tag{4.33}$$

by taking the weak coupling limit ($\Omega_1\gg\Gamma$).

In other words, if the coupling is weak enough, the only effect of the thermal reservoir is to renormalise the frequency of the oscillator, whose ground state is then described by a wavefunction, as for the free oscillator, yielding Markovian behaviour in the framework of stochastic mechanics.

This is not in contrast with the absence of a Markovian limit at $T=0$ in the usual quantum-mechanical treatment of the Brownian oscillator (Ullersma 1966), in which case the inequality $\hbar\Gamma\ll kT$ must be satisfied for Markovianity. In order to understand the difference between the two cases it is enough to consider the limit of vanishing coupling with the thermal bath ($\Gamma=0$): stochastic mechanics then yields the Nelson Markov process for the ground state of the free oscillator with an exponentially decaying correlation function, while in quantum mechanics one obtains an oscillating correlation function.

As a last comment, carrying out the integral (4.33) one obtains

$$S_\xi(t)=\frac{\hbar}{2\pi\Omega}\,\mathrm{Im}\{\exp[\mathrm{i}(\Gamma+\mathrm{i}\Omega)t]\,\mathrm{Ei}[-\mathrm{i}(\Gamma+\mathrm{i}\Omega)t] + \exp[-\mathrm{i}(\Gamma+\mathrm{i}\Omega)t]\,\mathrm{Ei}[\mathrm{i}(\Gamma+\mathrm{i}\Omega)t]\} \tag{4.34}$$

where Ei( ) is the exponential integral function and Ω is defined by $\Omega_1^2=\Omega^2+\Gamma^2$. This expression yields, as should be expected, the Euclidean version of the long-time tail

$$S_\xi(t)\to\frac{2\hbar\Gamma}{\pi\Omega_1^4}\frac{1}{t^2} \tag{4.35}$$

previously found by other authors (Grabert *et al* 1984, Haake and Reibold 1985, Riseborough *et al* 1985).

In conclusion we have derived the exact random process $x_0(t)$ performed by the Brownian oscillator when the thermal bath is described by the microscopic model of Ullersma. The additive structure (4.1) of this process allows us to identify the thermal component $q_0(t)$ and the quantum component $\xi_0(t)$ of the fluctuations. The thermal component prevails at high temperature and carries the dissipative effect of the thermal bath. The quantum component prevails at low temperature and displays the interesting features arising at $T=0$, when the interaction with the bath cannot produce thermal fluctuations. The drastic change in the nature of the fluctuations, as the temperature is lowered, is clearly illustrated by the fact that the Brownian oscillator attempts to behave, at zero temperature, as closely as possible to a particle in a pure quantum state by sharing overall properties characteristic of quantum fluctuations. Clearly the similarity cannot be complete, since the bath also plays a role at zero temperature, producing non-Markovian behaviour. This has been investigated in detail by deriving the equation of motion of the zero-temperature process, which is a quite interesting result in the context of stochastic mechanics since it provides the first example of a non-Markovian Nelson process.

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